

Geometry I, 2006 : Coursework 8 Solutions

1. Let A be an $n \times n$ matrix.

(i) [1 mark] A is *invertible* if there is some $n \times n$ matrix B such that $AB = BA = I_n$.

[1 mark] If A is invertible then the $n \times n$ matrix B such that $AB = BA = I_n$ is called the *inverse* of A .

(ii) [1 mark each for a completely correct answer]

(a) Let $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Then $\det(A) = 2$, and so A is invertible. $A^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$.

(b) Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$. Then $\det(A) = -1$, and so A is invertible. $A^{-1} = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$.

(c) Let $A = \begin{pmatrix} 2 & -3 \\ 6 & -9 \end{pmatrix}$. Then $\det(A) = 0$, and so A is not invertible.

2. [2 marks each; 1 mark if answer correct, but justification only partly correct; no marks if wrong answer or no justification or wrong justification.]

(i) This is true, for if A is an invertible $n \times n$ matrix and α is a non-zero scalar, then

$$(\alpha A)((1/\alpha)A^{-1}) = (\alpha/\alpha)(AA^{-1}) = I_n,$$

and

$$((1/\alpha)A^{-1})(\alpha A) = (\alpha/\alpha)(A^{-1}A) = I_n,$$

and so αA is invertible (having inverse $(1/\alpha)A^{-1}$).

(ii) This is false. Suppose $A = I_n$, $B = -I_n$. Then A and B are invertible (A and B are each their own inverse), but $A + B = 0_{nn}$, which is not invertible.

(iii) This is true, for if $AB = AC$ and A is invertible, we have $A^{-1}AB = A^{-1}AC$, and so $I_n B = I_n C$, $B = C$.

(iv) This is true, for if A is an invertible 2×2 matrix, we have

$$\det(A) \det(A^{-1}) = \det(AA^{-1}) = \det(I_2) = 1,$$

and so $\det(A) \neq 0$ and $\det(A^{-1}) = 1/\det(A)$.

3. (i) [1 mark] $\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} = u_1 v_2 - v_1 u_2 = u_1 v_2 - u_2 v_1 = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}.$

(ii) [2 marks] $\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = u_1 \begin{vmatrix} v_2 & w_2 \\ v_3 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & w_1 \\ v_3 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix}$
 $= u_1(v_2 w_3 - w_2 v_3) - u_2(v_1 w_3 - w_1 v_3) + u_3(v_1 w_2 - w_1 v_2)$
 $= u_1 v_2 w_3 - u_1 w_2 v_3 - u_2 v_1 w_3 + u_2 w_1 v_3 + u_3 v_1 w_2 - u_3 w_1 v_2$
 $= u_1 v_2 w_3 - u_1 v_3 w_2 - v_1 u_2 w_3 + v_1 u_3 w_2 + w_1 u_2 v_3 - w_1 u_3 v_2$
 $= u_1(v_2 w_3 - v_3 w_2) - v_1(u_2 w_3 - u_3 w_2) + w_1(u_2 v_3 - u_3 v_2)$
 $= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - v_1 \begin{vmatrix} u_2 & u_3 \\ w_2 & w_3 \end{vmatrix} + w_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$

4. [1 mark each; some reasonable calculation must be shown (together with the right answer) to get a mark.]

(i) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 3 & 1 \end{vmatrix} = 5 + 2 - 3 = 4;$

(ii) $\begin{vmatrix} 1 & 1 & 1234 \\ 1 & 2 & -1234 \\ 1 & 3 & 1234 \end{vmatrix} = 1234 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 3 & 1 \end{vmatrix}$

(since multiplying a column by a scalar multiplies the determinant by that scalar)
 $= 1234(4) = 4936;$

(iii) $\begin{vmatrix} 27 & 1234 & 1234 \\ 19 & 1234 & 1234 \\ -15 & 1234 & 1235 \end{vmatrix} = \begin{vmatrix} 27 & 1234 & 0 \\ 19 & 1234 & 0 \\ -15 & 1234 & 1 \end{vmatrix}$

(subtracting column 2 from column 3 does not change the determinant)

$$= \begin{vmatrix} 0 & 27 & 1234 \\ 0 & 19 & 1234 \\ 1 & -15 & 1234 \end{vmatrix}$$

(cycling around the three columns does not change the determinant)

$$= 1234 \begin{vmatrix} 0 & 27 & 1 \\ 0 & 19 & 1 \\ 1 & -15 & 1 \end{vmatrix} = 1234(27 - 19) = 1234(8) = 9872;$$

(iv) $\begin{vmatrix} 100 & 100 & 100 \\ -10 & -10 & -10 \\ 567 & 101 & -100 \end{vmatrix} = \begin{vmatrix} 100 & -10 & 567 \\ 100 & -10 & 101 \\ 100 & -10 & -100 \end{vmatrix}$ (by exercise 3(ii))

$$= \begin{vmatrix} 0 & -10 & 567 \\ 0 & -10 & 101 \\ 0 & -10 & -100 \end{vmatrix} = 0.$$

(adding 10 times column 2 to column 1 does not change the determinant)